Derivation and Applications of Wien's Displacement Law

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In this paper, I will briefly derive Wien's Displacement Law and discuss its applications in Astrophysics. Specifically, I will discuss the history of Wien's Displacement Law, including how it was developed and original uses. I will then derive Wien's Displacement Law from Planck's Law. I will show how Wien's Displacement Law can be used to derive the temperature of stars from stellar spectra observations. Finally, I will present the limitations of Wien's Displacement Law.

INTRODUCTION

Wien's Displacement Law states that the black-body radiation curve for an object varies with temperature. Specifically, Wien's Displacement Law describes how the peak wavelength of black-body radiation changes with temperature. Wien's Displacement Law was orginally formulated by Wilhelm Wien in 1893.

Wien used a thermodynamic thought experiment to derive his law. Wien considered a cavity with light inside slowly expanding in an adiabatic way. He found that the frequency and energy of the light change in the same way. Based on this, Wien theorized that the peak wavelength must change in accordance with the change in energy, which is directly related to the temperature. [1]

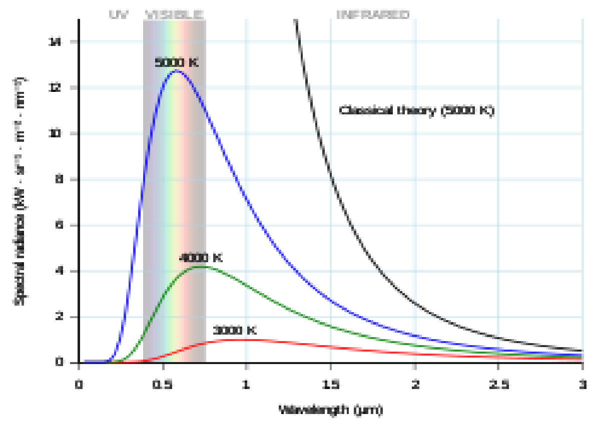
Seven years later in 1900, Max Planck derived Planck's Law, which describes the spectral density of electromagnetic radiation from a black-body, formulated as:

$$B(\lambda, T) = \frac{2hc^2}{\lambda^5}\frac{1}{e^{\frac{hc}{\lambda k_B T}} - 1} \tag{1}$$

Planck's Law can be seen as a more comprehensive and fundamentally sound version of Wien's Law that produces an entire function rather than just a peak wavelength. Considering this, deriving Wien's Law from Planck's Law is the most physically sound derivation for Wien's Law that we can produce. [2]

DERIVATION

To start with the derivation, we must understand Planck's Law. Planck's Law produces a continuous function that is unique to each black-body temperature. The graph below shows the black-body spectrum at various temperatures, as predicted by Planck's Law.



Wien's Law determines the wavelength of peak emission, so deriving Wien's Law involves finding the maximum value of Planck's Law as a function of temperature.

The first step is to take the partial derivative of Planck's Law (1) with respect to wavelength, λ.

$$\frac{\partial B}{\partial \lambda} = \frac{2hc^2}{\lambda^6\left(e^{\frac{hc}{\lambda k_B T}} - 1\right)}\left(\frac{e^{\frac{hc}{\lambda k_B T}}}{e^{\frac{hc}{\lambda k_B T}} - 1} - 5\right) \tag{2}$$

Next, setting (2) equal to zero and simplifying:

$$\frac{hc}{\lambda k_B T}\frac{e^{\frac{hc}{\lambda k_B T}}}{e^{\frac{hc}{\lambda k_B T}-1}} - 5 = 0 \tag{3}$$

Defining $x \equiv \frac{hc}{\lambda k_B T}$, equation (3) becomes:

$$\frac{xe^x}{e^x - 1} - 5 = 0 \tag{4}$$

Rearranging equation (4) gives:

$$e^x\left(x - 5\right) + 5 = 0 \tag{5}$$

Equation (5) can be solved for x numerically using Newton's Method, resulting in $x = 4.965114$. We can now make a relationship for peak wavelength, λ_{peak}, in terms of temperature.

$$\lambda_{peak} = \frac{hc}{xk_BT} \tag{6}$$

Plugging in the constant values $h = 6.626 \times 10^{-27}, c = 3 \times 10^{10}, k_B = 1.381 \times 10^{-16}$ and the value of x found in the previous part, gives Wien's Law:

$$\lambda_{peak} = \frac{b}{T} \tag{7}$$

where $b = 2.897\,77 \times 10^{-3}\,\mathrm{m\,K}$, known as the Wien Displacement Constant. Wien originally found this constant experimentally, and it was later analytically supported by Planck.

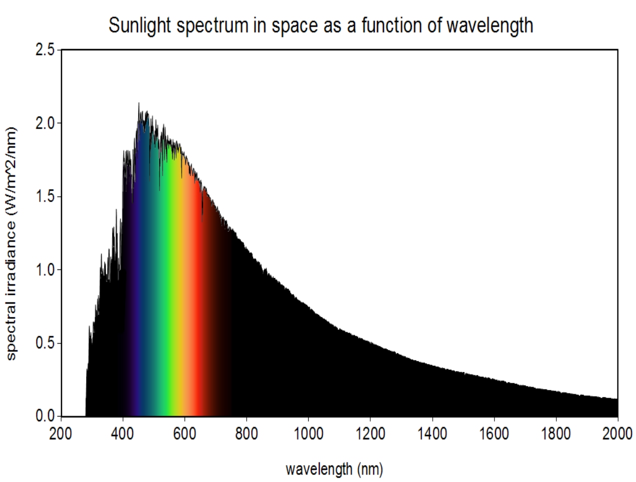
APPLICATIONS

Wien's Displacement Law is widely used as a method to determine the temperature of a black-body from its spectra. For this purpose, we can rearrange equation (7) to make calculations easier.

$$T = \frac{b}{\lambda_{peak}} \tag{8}$$

Applications in Astronomy

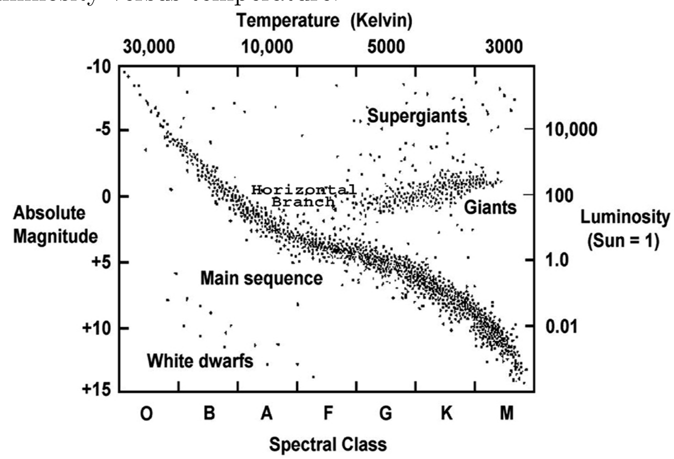
In Astronomy, Wien's Displacement Law is used to estimate the temperatures of stars. For example, let us consider the Sun. The stellar spectrum of the Sun is shown below.



This graph was produced by doing spectroscopy on Sunlight. From the figure, we can see that the peak wavelength is approximately $\lambda_{peak} = 5.000 \times 10^{-7}$ m. Now, plugging this into equation (8):

$$T_{Sun} \approx \frac{2.89777}{5.0 \times 10^{-7}} \approx 5800K \tag{9}$$

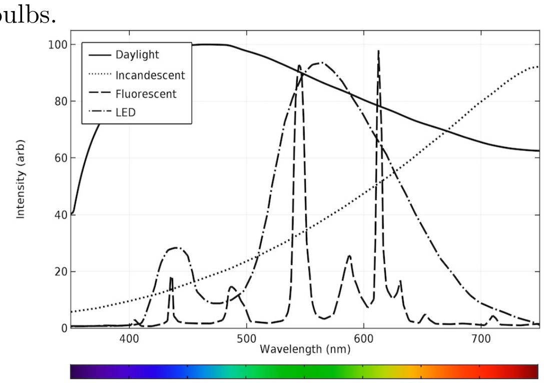
Understanding the connection between temperature and spectra of stars helps us gain a fundamental understanding of stars. Wien's Law later helped the development of stellar categorization. A key example of this is the Herzsprung-Russell Diagram, which creates a plot of luminosity versus temperature.



As shown in the diagram, categorizing stars becomes fairly easy when you have this relationship between luminosity and temperature. Further, this knowledge allows us to investigate the evolution of stars, from the main sequence to giants and ultimately to white dwarfs.

Everyday Application - Light bulbs

An everyday application of Wien's Law is in light bulbs. The wavelength that a light bulb produces is an exceedingly important quality of it. The closer a light bulb's spectra is to that of daylight, the better. The graphic below shows the spectra of several types of light bulbs.

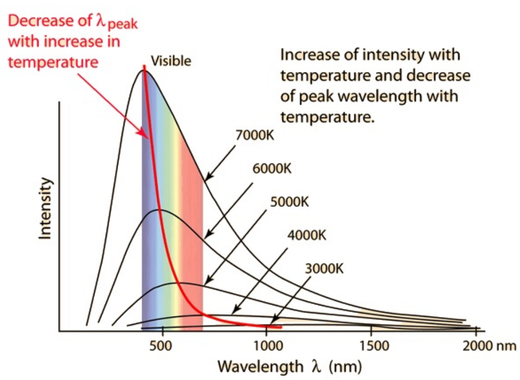


As we have seen, Wien's Law shows us that the temperature of the emitter is related to the spectrum that is produced. Therefore, the temperature of the filament determines the peak wavelength and spectra we observe for incandescent light bulbs. The best incandescent light bulbs have filament that can heat to a temperature that produces a spectrum that is closest to daylight. Further, higher quality incandescent light bulbs have a filament that take longer to burn out. As a filament gets older or is of poor quality, it will not be able to heat up to as high of temperatures, which produces a redder peak wavelength.

Light bulb technology has advanced and now the light produced by light bulbs is no longer dependent on the temperature of a filament. As a result, modern light bulbs have the capability to produce spectra that is not bound to Wien's Displacement Law and Planck's Law. This can be seen on the graphic, where the spectra for LED light does not follow a curve governed by Planck's Law - it has several peaks and local minimum, which would not happen under Planck's Law. This freedom can be used to make a light bulb that produces light that mimics daylight without having to burn filament at the temperature of the Sun.[3]

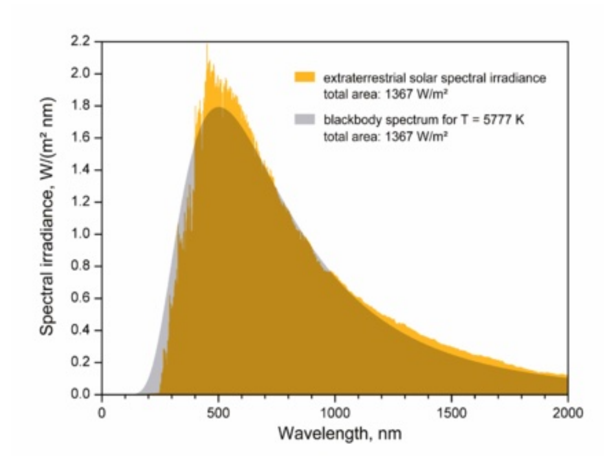
LIMITATIONS

Despite its usefulness in many situations, Wien's Displacement Law does not make accurate predictions at long wavelengths and low temperatures. A continuous curve cannot be obtained from Wien's Law at low temperatures.



As you can see in the figure, the graph of peak wavelength begins to asymptote as the the wavelength increases and the temperature decreases. Further, the functions produced by Planck's Law (represented in black, at various temperatures) broadens as the temperature increases.

Another limitation is that we must assume that the emitter is a black-body. In reality, no object can be a true black-body, so the closest an object can get is emitting as a near black-body. This leads to some slightly inaccurate predictions from Wien's Displacement Law and Planck's Law, but they both generally work well for making approximations. The graphic below shows the difference between the true Sunlight spectrum and the spectrum predicted for a black-body by Planck's Law.



Alternative Methods to Determine Temperature

Wien's Displacement Law is not the only way to estimate the temperature of an object that emits radiation. One of the best ways to measure temperature at a distance is by using infrared thermometers. However, this method is limited to use at close distances.

In astronomy, there are other mathematical methods to try to determine the temperature of stars. A very common one is the Stefan-Boltzmann Law, which is formulated as:

$$P = \epsilon \sigma A T^4 \tag{10}$$

where P is the power radiated, which is equal to the luminosity. A is the area of the emitter ($A = 4\pi R^2$ for stars - we can consider them as spherical). As for the other constants, ϵ is the emissivity ($\epsilon = 1$ for an ideal black-body) and σ is the Stefan-Boltzmann constant ($\sigma = 5.6704 \times 10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}}$).

CONCLUSIONS

I presented Wien's Displacement Law and its applications in this paper. Wien's Law can be derived from Planck's Law, which produces a function of wavelength and intensity for a black-body of variable temperature. As shown in the paper, Wien's Law provides an estimate for the wavelength of peak intensity produced by a black-body radiating at some temperature.

Despite the fact that Planck's Law was formulated after Wien's Law, it provides a more fundamental understanding of the relationship between wavelength, intensity, and temperature. Due to this, Planck's Law is appropriate to be used to derive Wien's Law. As a part of deriving Wien's Law, we derived the Wien Displacement Constant.

From there, we explored the applications of Wien's Law, especially deriving stellar temperatures. We used the stellar spectra of the Sun to estimate its temperature. Finally, I presented the limitations of Wien's Law, which is at long wavelenghts and low temperatures.

ACKNOWLEDGMENTS

I would like to thank Professor Marka for his generous help and guidance in my study of physics and learning to write in Latex. Further, I would also like to thank Professor Agüeros for helping me understand Wien's Displacement Law and offering some historical context on its derivation. Finally, I want to take this opportunity to thank all of my teachers for inspiring me to push boundaries, explore, and discover.

[1] "Wiens displacement law," http://snst-hu.lzu.edu.cn/zhangyi/ndata/Wien's_Displacement_Law.html ().
[2] "Planck's law," http://rossby.msrc.sunysb.edu/~marat/MAR542/ATM542-Chapter2.pdf ().
[3] "Ucsc physics demonstration room," http://ucscphysicsdemo.sites.ucsc.edu/physics-5b6b-demosopticswiens-law ().